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# The Sugawara Model in General Relativity. A. The Case for Three Current Vectors ($SU(2)$)

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*Abstract*

A general method of solving the equations of Sugawara's field theory of currents has been developed, and illustrated by applying it to the set of three currents. These are inserted into Einstein's field equations which have been solved together with the co-variant 'gauge' conditions for a gravitational field involving cylindrical symmetry. A further transformation exhibits the triad formed by the current vectors and exhibits clearly the deviations of the line-element from Schwarzschild's exterior solution. In a subsequent paper the case for eight vector currents corresponding to $SU(3)$ will be treated in similar fashion.

## 1. *Introduction*

One of the outstanding problems in general relativity is to introduce the interaction of elementary particles with the gravitational field, and, in particular, to obtain an equation of state resulting from that interaction. This would give us a better insight into processes which are responsible for the presence of gravitational fields, and which usually are described heuristically by a pressure and density as well as an equation of state connecting them (Wyman, 1949). On the other hand, if we do know the nature of the interaction, and, in particular, the form of the energy momentum tensor a solution of the co-determined problem will give us the gravitational field as well.

The abundance of different types of hadrons (Jackson, 1958) makes it unrealistic to consider all possible interactions in detail, and one must be satisfied with a model which gives the main feature of the strong interaction. The many successful predictions of the so-called 'eight-fold way' (Gell-Mann & Ne'eman, 1964) and its underlying group structure is a good indication that a model based on it might, at least for the beginning, be sufficient to describe the desired interaction. A possible model is provided by Sugawara's field theory of currents (Sugawara, 1968) in which the field is represented by a number of currents $A_\mu{}^i$ ($i = 1$ to 8) and the corresponding axial vectors $V_\mu{}^i$. Neglecting the axial vector currents, it has been shown† that these satisfy the equations

$$A^i_{\mu,\nu} - A^i_{\nu,\mu} = \frac{1}{K}\delta^{im} f_{mjk} A_\mu{}^j A_\nu{}^k \tag{1.1}$$

where we have limited us to a $c$-number theory. The $f_{ijk}$ are the completely antisymmetrical structure constants (Gell-Mann & Ne'eman, 1964, Table II) and $K$ is the coupling constant. (The Kronecker deltas $\delta^{im}$ have been introduced to preserve the explicit covariance of the equations.) As usual, commas denote partial derivatives with respect to the co-ordinates $x^\mu$. (Since the equation is antisymmetric upon interchange of $\mu$ and $\nu$ it is not necessary to replace partial derivatives by covariant ones.)

In order to illustrate the method and to simplify matters we shall limit ourselves for the present to three currents $A_\mu{}^i$ ($i = 1, 2, 3$).

$$A^i_{\mu,\nu} - A^i_{\nu,\mu} = \frac{1}{K}\delta^{im} \epsilon_{mjk} A_\mu{}^j A_\nu{}^k \tag{1.2}$$

where $\epsilon_{ijk}$ are the components of the Levi-Civita tensor.

In addition to the field equations the vector currents also must satisfy the 'gauge' condition, whose covariant form is given by

$$D_\mu A^{\mu i} = 0 \tag{1.3}$$

for any value of $i$ and where $D_\mu$ denotes covariant differentiation

Following Sugawara (1968) and limiting ourselves to a $c$-number theory for which one can assume commutability the energy momentum tensor has the simple form

$$T_{\mu\nu} = -\frac{1}{K}\left[A_\mu{}^i A_{\nu i} - \tfrac{1}{2} g_{\mu\nu}(A_{\alpha i} A^{\alpha i})\right] \tag{1.4}$$

where summation over repeated indices is implied.‡ It is this energy momentum tensor which has to be introduced into Einstein's field equations

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \tag{1.5}$$

to provide us with a co-determined solution.

If we write (1.5) in its equivalent form

$$R_{\mu\nu} = -\kappa\,(T_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} T) \tag{1.6}$$

it follows from (1.4) that the Einstein equations reduce to

$$R_{\mu\nu} = -\kappa A_{\nu i} A_{\nu j} \delta^{ij}$$

which is easier to handle mathematically.

† The fact that we limit ourselves to a $c$-number theory makes it possible to omit terms of the form $A_\mu{}^k A_\mu{}^j$.

‡ We adopt the usual summation convention both for the index $\mu = 1, 2, 3, 4$ and vector currents $i = 1, 2, 3$.

The present problem, then, consists in finding solutions of the field equations (1.2) and satisfying the gravitational field equations (1.6) subject to the subsidiary conditions (1.3). In Section 2 we shall develop a method for the solution of the field equations and illustrate it by applying it to the simpler system (1.2). This will be followed in Section 3 by explicit solutions of the coupled system (1.2), (1.6) satisfying the subsidiary conditions (1.3) for a simple case involving cylindrical symmetry.

The general case (1.1) will be treated in a subsequent paper in this series, where physically meaningful solutions of the field equations (1.1), (1.6) satisfying the subsidiary conditions (1.3) are also exhibited.

## 2. *Field Equations for SU(2)*

The problem is to solve the equations (1.2) which can be written in full also as

$$A^i_{\mu,\nu} - A^i_{\nu,\mu} = \frac{1}{\kappa}(A_\mu{}^j A_\nu{}^k - A_\nu{}^j A_\mu{}^k)$$
$$(i, j, k = 1, 2, 3 \text{ in cyclic order}) \tag{2.1}$$

If we now set

$$\omega^i = A_\mu{}^i \, dx^\mu \qquad (i = 1, 2, 3) \tag{2.2}$$

then, on account of

$$\omega^j \Lambda \omega^k = (A_\mu{}^i A_\nu{}^k - A_\nu{}^j A_\mu{}^k)\, dx^\mu \Lambda\, dx^\nu$$

and

$$d\omega^i = \frac{\partial A_\mu{}^i}{\partial x^\nu} dx^\mu \Lambda\, dx^\nu$$

(2.1) becomes

$$d\omega^i = \frac{1}{\kappa}\, \omega^j \Lambda \omega^k \qquad (i, j, k = 1, 2, 3 \text{ in cyclic order}) \tag{2.3}$$

where $\Lambda$ denotes the outer product. The factor $1/K$ can be absorbed by setting $\omega^i = Kw^i$, so that we have

$$dw^i = w^j \Lambda w^k$$
$$= \tfrac{1}{2}\epsilon^i{}_{jk} w^j w^k \tag{2.3'}$$

We now introduce the matrix

$$\mathbf{\Omega} = \| w_{ij} \| \tag{2.4}$$

related to the vector $w^i$ through

$$w^i = \tfrac{1}{2}\epsilon^{ijk} w_{jk} \qquad \text{or} \qquad w_{ij} = \epsilon_{ijk} w^k$$

equation (2.3) can be written as (Flanders, 1963)

$$d\mathbf{\Omega} = \mathbf{\Omega}^2 \tag{2.5}$$

A proof of this result, which will also be useful for our further work, is given in Appendix A. To find a (general) solution of (2.5) let

$$\mathbf{\Omega} = d\mathbf{R}\mathbf{R}^{-1} \tag{2.6}$$

where $\mathbf{R} = \|R_{ij}\|$ is an arbitrary non-singular square matrix. Also, let

$$\mathbf{\Theta} = d\mathbf{\Omega} - \mathbf{\Omega}^2$$

From (2.6) we have

$$d\mathbf{R} = \mathbf{\Omega}\mathbf{R}$$

Hence

$$d(d\mathbf{R}) = d\mathbf{\Omega}\mathbf{R} - \mathbf{\Omega}d\mathbf{R} = (\mathbf{\Theta} + \mathbf{\Omega}^2)\mathbf{R} - \mathbf{\Omega}(\mathbf{\Omega}\mathbf{R}) = \mathbf{\Theta}\mathbf{R} = 0$$

It follows, therefore, that $\mathbf{\Theta} = 0$ since $\mathbf{R}$ is arbitrary. However, this implies that (2.5) holds so that (2.6) is the required solution. In particular, it can be shown that $\mathbf{R}$ is orthogonal if $\mathbf{\Omega}$ is skew-symmetric.

Let us now return to (2.3). From (2.6) and the relation between the vectors $w^i$ and the matrix $\mathbf{\Omega} = \|w_{ij}\|$ it follows that

$$\begin{aligned} w^k &= \tfrac{1}{2}\epsilon^{kij}w_{ij} = \tfrac{1}{2}\sum_m \epsilon^{kij}\, dR_{mi}R_{mj} \\ &= \tfrac{1}{2}\sum_m \epsilon^{kij}\frac{\partial R_{mi}}{\partial x^\mu}R_{mj}\, dx^\mu \end{aligned} \tag{2.7}$$

Equating the two expressions (2.2) and (2.7) then yields

$$A_\mu{}^k = \frac{K}{2}\epsilon^{kij}\sum_m \frac{\partial R_{mi}}{\partial x^\mu}R_{mj} = -\frac{K}{2}\sum_m \epsilon^{kij}\frac{\partial R_{mj}}{\partial x^\mu}R_{mi} \tag{2.8}$$

for the vectors $A_\mu{}^i$. Furthermore, since $\mathbf{R}$ is an orthogonal matrix, we can express it in terms of three unit vectors $\mathbf{R}_i$

$$\mathbf{R} = (\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3) \qquad \text{where } \mathbf{R}_i = \begin{pmatrix} R_{1i} \\ R_{2i} \\ R_{3i} \end{pmatrix} \tag{2.9}$$

satisfying the ortho-normality relations

$$\mathbf{R}_i . \mathbf{R}_j = \delta_{ij}$$

In terms of these vectors (2.8) can then be written as

$$A_\mu{}^k = \frac{K}{2}\epsilon^{kij}\frac{\partial \mathbf{R}_i}{\partial x^\mu}\mathbf{R}_j = -\frac{K}{2}\epsilon^{kij}\frac{\partial \mathbf{R}_j}{\partial x^\mu}\mathbf{R}_i \tag{2.10}$$

A particular representation of the vectors $\mathbf{R}_i$ is provided by the rotation group in terms of two angles $\alpha$ and $\beta$

$$R_1 = \begin{pmatrix} \sin\alpha\cos\beta \\ \sin\alpha\sin\beta \\ \cos\alpha \end{pmatrix}, \qquad R_2 = \begin{pmatrix} \cos\alpha\cos\beta \\ \cos\alpha\sin\beta \\ -\sin\alpha \end{pmatrix}, \qquad \mathbf{R}_3 = \begin{pmatrix} -\sin\beta \\ \cos\beta \\ 0 \end{pmatrix}$$

(The general case, of course, would be achieved by having recourse to the three Eulerian angles. However, we are interested in simple solutions for which two parameters will be sufficient.) Noting that

$$\frac{\partial \mathbf{R}_1}{\partial x^\mu} = \alpha_\mu \mathbf{R}_2 + \sin\alpha\beta_\mu \mathbf{R}_3$$

$$\frac{\partial \mathbf{R}_2}{\partial x^\mu} = -\alpha_\mu \mathbf{R}_1 + \cos\alpha\beta_\mu \mathbf{R}_3$$

$$\frac{\partial \mathbf{R}_3}{\partial x^\mu} = -\sin\alpha\beta_\mu \mathbf{R}_1 - \cos\alpha\beta_\mu \mathbf{R}_2$$

it follows immediately from (2.10) that a particular solution of our original problem (1.2) is given by

$$\begin{aligned} A_\mu{}^1 &= K\cos\alpha\beta_\mu \\ A_\mu{}^2 &= -K\sin\alpha\beta_\mu \\ A_\mu{}^3 &= K\alpha_\mu \end{aligned} \tag{2.11}$$

These vectors still have to satisfy the gauge conditions (1.3), but since these will depend on the metric chosen, we shall discuss them together with the field equations (1.6) in the next section.

## 3. *The Codetermined Problem*

We now turn to the subsidiary conditions (1.3). Inserting the values found for the vectors $A_\mu{}^i$ (2.11) it is not difficult to see that we have to solve the system of equations

$$\frac{\partial}{\partial x^\mu}(\alpha_\nu g^{\mu\nu}\sqrt{(-g)}) = \frac{\partial}{\partial x^\mu}(\beta_\nu g^{\mu\nu}\sqrt{(-g)}) = 0 \qquad \alpha_\mu\beta_\nu g^{\mu\nu} = 0 \tag{3.1}$$

Since these depend on the gravitational potentials it is important to use a proper line-element. Obviously, the simplest one would be the spherically symmetric line-element, but a simple consideration shows that at least two independent variables are necessary in order to avoid contradictions. However, if we use a line-element with cylindrical symmetry (Synge, 1960)

$$ds^2 = -e^{2(\nu-\tau)}(d\rho^2 + dz^2) - \rho^2 e^{-2\tau} d\phi^2 + e^{2\tau} dt^2$$
$$\nu = \nu(\rho, z), \qquad \tau = \tau(\rho, z) \tag{3.2}$$

it turns out that the effect of the gravitational field cancels out in (3.1).

Assuming that $\alpha$ and $\beta$ are only functions of $x^1 = \rho$ and $x^2 = z$ we find from (3.1) and (3.2) that

$$\alpha_{11} + \alpha_{22} + \frac{\alpha_1}{\rho} = 0, \qquad \beta_{11} + \beta_{22} + \frac{\beta_1}{\rho} = 0 \tag{3.3}$$

together with

$$\alpha_1\beta_1 + \alpha_2\beta_2 = 0 \tag{3.3$'$}$$

where as usual subscripts denote differentiation with respect to $x^1$ and $x^2$ respectively. These are Laplace's equations in cylindrical coordinates $(\rho, \phi, z)$ in ordinary Euclidean space for a function independent of $\phi$, for which a solution can be found easily.

On the other hand, from (1.6) we obtain the system of equations

$$R_{11} + R_{22} = 2\left[\Delta\nu - \left(\Delta\tau + \frac{\tau_1}{\rho}\right) + {\tau_1}^2 + {\tau_2}^2\right]$$

$$= -\kappa K^2 [({\alpha_1}^2 + {\alpha_2}^2) + ({\beta_1}^2 + {\beta_2}^2)] \tag{3.4a}$$

$$R_{11} - R_{22} = 2\left[{\tau_1}^2 - {\tau_2}^2 - \frac{\nu_1}{\rho}\right] = -\kappa K^2 [({\alpha_1}^2 - {\alpha_2}^2) + ({\beta_1}^2 - {\beta_2}^2)] \tag{3.4b}$$

$$R_{12} = 2\tau_1\tau_2 - \frac{\nu_2}{\rho} = -\kappa K^2 [\alpha_1\alpha_2 + \beta_1\beta_2] \tag{3.4c}$$

$$R_{33} = R_{44} = 0 \tag{3.4d}$$

where we have defined

$$\Delta\tau = \tau_{11} + \tau_{22} \qquad \text{and} \qquad \Delta\nu = \nu_{11} + \nu_{22}$$

As has been shown by Synge (1960) (3.4d) leads again to a Laplace equation for $\tau$ of the form (3.3)

$$\Delta\tau + \frac{\tau_1}{\rho} = 0 \tag{3.5}$$

The two equations (3.4b) and (3.4c) can be solved for $\nu_1$ and $\nu_2$ respectively. Inserting these values into (3.4a) and making use of (3.5) and (3.3) it can be shown that (3.4a) is identically satisfied, as well as the compatibility relation

$$\nu_{21} = \nu_{12}$$

We therefore find an explicit solution to the codetermined problem provided we take any solutions of the Laplace equations (3.3) satisfying the orthogonality conditions (3.3′) together with a solution of (3.5).

At this point it is advantageous to introduce the coordinates $\lambda$, $\mu$ used by

Erez & Rosen (1959) to describe the gravitational field of a particle possessing a multipole moment.

$$\rho = m(\lambda^2 - 1)^{1/2}(1 - \mu^2)^{1/2}$$
$$z = m\lambda\mu \qquad \lambda \geqslant 1$$
$$-1 \leqslant \mu \leqslant 1 \tag{3.6}$$

Here $m$ denotes the mass of the particle, but as far as we are concerned it is just an additional constant. In terms of these coordinates (3.3) becomes

$$[(\lambda^2 - 1)\alpha_\lambda]_{,\lambda} + [(1 - \mu^2)\alpha_\mu]_{,\mu} = 0 \tag{3.7}$$

with a similar equation for $\beta$, where subscripts $\lambda$ and $\mu$ denote derivatives with respect to these variables. Furthermore, the orthogonality condition (3.3′) is now

$$(\lambda^2 - 1)\alpha_\lambda\beta_\lambda + (1 - \mu^2)\alpha_\mu\beta_\mu = 0 \tag{3.7′}$$

A simple way of satisfying this condition is to assume that

$$\alpha = \alpha(\lambda) \qquad \text{and} \qquad \beta = \beta(\mu) \tag{3.8}$$

(We could, of course, have made a similar assumption in the original coordinates but it is physically more satisfactory not to assume that $\beta(z)$ in order to avoid singularities along the $z$-axis.)

Inserting (3.8) into (3.7) then gives

$$\alpha_\lambda = \frac{A}{\lambda^2 - 1} \qquad \text{or} \qquad \alpha = \frac{A}{2}\ln\frac{\lambda - 1}{\lambda - 1}$$
$$\beta_\mu = \frac{B}{1 - \mu^2} \qquad \qquad \beta = \frac{B}{2}\ln\frac{1 + \mu}{1 - \mu} \tag{3.9}$$

where $A$ and $B$ are two constants.

In terms of the new variables it is again possible to solve (3.4b) and (3.4c) for $\nu_\lambda$ and $\nu_\mu$ with the result

$$\nu_\lambda = \frac{1 - \mu^2}{2(\lambda^2 - \mu^2)}\{[(\lambda^2 - 1)(2\tau_\lambda{}^2 + \kappa K^2\alpha_\lambda{}^2) - (1 - \mu^2)(2\tau_\mu{}^2 + \kappa K^2\beta_\mu{}^2)]\lambda$$
$$- 4\mu(\lambda^2 - 1)\tau_\lambda\tau_\mu\} \tag{3.10a}$$

$$\nu_\mu = \frac{\lambda^2 - 1}{2(\lambda^2 - \mu^2)}\{[(\lambda^2 - 1)(2\tau_\lambda{}^2 + \kappa K^2\alpha_\lambda{}^2) - (1 - \mu^2)(2\tau_\mu{}^2 + \kappa K^2\beta_\mu{}^2]\mu$$
$$+ 4\lambda(1 - \mu^2)\tau_\lambda\tau_\mu\} \tag{3.10b}$$

Just as before it can be shown that (3.4a) and the compatibility conditions

$$\nu_{\lambda\mu} = \nu_{\mu\lambda}$$

are identically satisfied.

Similarly, the equation for $\tau$ (3.5) is given by

$$[(\lambda^2 - 1)\tau_\lambda]_{,\lambda} + [(1 - \mu^2)\tau_\mu]_{,\mu} = 0 \tag{3.11}$$

It has been shown by Erez & Rosen (1959) that a family of solutions can be obtained by separation of variables. A general and well-behaved solution is given by

$$\tau = \sum_l a_l P_l(\mu) Q_l(\lambda) \tag{3.12}$$

where $P_l(\mu)$ and $Q_l(\lambda)$ are the Legendre polynomials of the first and second kind respectively. Here $l = 0, 1, 2, \ldots$ give the various contributions of multipoles of order $l$. We limit ourselves for simplicity to the lowest order (i.e. setting the separation constant equal to zero).

$$\tau = \frac{L}{2} \ln \frac{\lambda - 1}{\lambda - 1} \tag{3.12$'$}$$

Finally, inserting the values found for $\alpha_\lambda, \beta_\mu$, and $\tau_\lambda$ into (3.10), we obtain

$$\nu = \frac{a}{4} \ln \frac{\lambda^2 - 1}{\lambda^2 - \mu^2} + \frac{b}{4} \ln \frac{1 - \mu^2}{\lambda^2 - \mu^2} \tag{3.13}$$

where $a$ and $b$ are constants related to our previous constants through

$$\begin{aligned} a &= 2L^2 + \kappa K^2 A^2 \\ b &= \qquad\quad \kappa K^2 B^2 \end{aligned} \tag{3.13$'$}$$

It is of interest to carry out a further transformation involving both a conformal mapping of space and a coordinate transformation resulting in

$$\lambda = (r/m) - 1 \qquad \text{and} \qquad \mu = \cos\theta \tag{3.14}$$

† A more general solution of (3.11) but still of lowest order would be

$$\tau = \left(\frac{L}{2} \ln \frac{\lambda - 1}{\lambda + 1}\right)\left(\frac{M}{2} \ln \frac{1 + \mu}{1 - \mu}\right)$$

together with

$$\nu = \frac{a}{4} \ln \frac{\lambda^2 - 1}{\lambda^2 - \mu^2} + \frac{b}{4} \ln \frac{1 - \mu^2}{\lambda^2 - \mu^2} - c \ln \frac{\lambda - \mu}{\lambda - \mu}$$

where now

$$L = 2M^2 + K^2 B^2, \qquad c = LM$$

where $r$ and $\theta$ may be interpreted as the radius vector and azimuth angle of a spherical polar system of coordinates. In terms of these coordinates we find from (3.9)

$$\alpha = \frac{A}{2} \ln \left(1 - \frac{2m}{r}\right)$$
$$\beta = \frac{B}{2} \ln \frac{1 + \cos\theta}{1 - \cos\theta} \tag{3.15}$$

If we also transform the vectors $A_\mu{}^i$ according to (3.14) we obtain

$$\begin{aligned} &A_r{}^1 = 0 && A_\theta{}^1 = B \sin\alpha/\sin\theta \\ &A_r{}^2 = 0 && A_\theta{}^2 = B \cos\alpha/\sin\theta \\ &A_r{}^3 = \alpha_r = \frac{A}{m}\left(\frac{r}{m}\right)^{-2}\left(1 - \frac{2m}{r}\right)^{-1}, && A_\theta{}^3 = 0 \end{aligned} \tag{3.16}$$

which shows clearly the triad formed by these vectors.

In these coordinates the line-element becomes†

$$\begin{aligned} ds^2 = \left(1 - \frac{2m}{r}\right)^L dt^2 - \left(1 - \frac{2m}{r}\right)^{(a/2)-L} \left(\frac{m \sin\theta}{r}\right)^b \left(\frac{dr^2}{1 - \frac{2m}{r}} + r^2\, d\theta^2\right) \\ - r^2 \sin\theta \left(1 - \frac{2m}{r}\right)^{1-L} d\phi \end{aligned} \tag{3.17}$$

Apart from the multiplying factors (3.17) is of the form of the Schwarzschild line element to which it actually reduces for $L = 1, a = 2$ and $b = 0$. (From the definition of these values (4.13′) it is seen that this corresponds to the free field core with $A = B = 0$.)

## 4. *Conclusion*

The above example illustrates the influence of the vector currents on the gravitational field. Even for the simple solution chosen, there is a significant deviation from the Schwarzschild field. Clearly, if one takes the more general solution (3.12) the deviation would be more pronounced. Also, we limited ourselves to a representation depending only on two angles, $\alpha$ and $\beta$, while in

† For the solution given in the previous footnote (p. 182) the line element (3.17) acquires additional factors. In particular, $L$ is to be replaced by

$$L \to \tfrac{1}{2} c \tanh^{-1} \cos\theta$$

and the second term to be multiplied by

$$\left(\frac{1 - \frac{2m}{r}\sin\frac{2\theta}{2}}{1 - \frac{2m}{r}\cos\frac{2\theta}{2}}\right)^{c/2}$$

general three angles could be introduced. This, in effect, would produce deviations depending not only on $r$ and $\theta$, but also $\phi$. Moreover, the particular form of the line-element used limited the effect, since the gauge conditions turned out to be independent of the gravitational field, which would not be true in general. For these reasons, it does not seem worthwhile to push the calculation further and try to derive some kind of effective equation of state, as was done, for example, in the case of a massive vector meson interaction with the gravitational field (Tauber, 1969). It can be seen from (1.4) and (3.16) that—at least in the present case—there exist simple relations between the components of the energy momentum tensor of the form

$$T_1{}^1 + T_2{}^2 = 0, \qquad T_3{}^3 = T_4{}^4$$

which could be used to derive an effective equation of state. (Of course, it is no longer true that the distribution can be characterised by a simple pressure and density, as is the case for spherical symmetric distributions of matter.)

The main purpose of the present calculation was to develop a method to solve the equations of Sugawara's field theory and to exhibit explicit solutions of the codetermined systems of equations satisfying the subsidiary conditions. The general case involving eight vectors will be treated in a subsequent paper in this series.

Finally, it is a pleasure to acknowledge discussions with Prof. H. Flanders who has suggested the particular method employed here.

*Appendix—The Matrix Equation for SU(2)*

The equation to be solved is (2.3′)

$$dw^i = \tfrac{1}{2}\epsilon^i_{jk} w^j w^k \qquad (i, j, k = 1, 2, 3) \tag{A.1}$$

Introducing the matrix (2.4) through

$$w_{ij} = \epsilon_{ijk} w^k$$

and since

$$\epsilon^{jmn}\epsilon_{kmn} = \delta_k{}^j, \qquad w^i = \tfrac{1}{2}\epsilon^{ijk} w_{jk} \tag{A.2}$$

we obtain

$$\sum_{k=1}^{3} w_{jk} w_{km} = \sum_{k=1} \epsilon_{ijk}\epsilon_{nkm} w^i w^n$$

or, since $w^i w^m$ is antisymmetric upon interchange of $i$ and $n$

$$\sum_{k=1}^{3} w_{jk} w_{km} = \tfrac{1}{2}\sum_{k=1} (\epsilon_{ijk}\epsilon_{nkm} - \epsilon_{njk}) w^i w^n$$

Consider now the Pauli matrices $\sigma$ which satisfy the commutation relations

$$[\boldsymbol{\sigma}_j, \boldsymbol{\sigma}_k] = 2i\epsilon_{jkm}\boldsymbol{\sigma}^m, \qquad \boldsymbol{\sigma}^m = \delta^{mn}\boldsymbol{\sigma}_n \tag{A.3}$$

and the Jacobi identity

$$[[\sigma_i,\sigma_i],\sigma_m] - [[\sigma_j,\sigma_m],\sigma_i] + [[\sigma_m,\sigma_i],\sigma_j] = 0 \tag{A.4}$$

Inserting (A.3) into (A.4) twice in succession results in the identity

$$\sum_{k=1}^{3} (\epsilon_{ijk}\epsilon_{nkm} - \epsilon_{njk}\epsilon_{ikm} + \epsilon_{njm}\epsilon_{kin}) = 0 \tag{A.5}$$

so that

$$\sum_{k=1}^{3} w_{jk}w_{km} = \tfrac{1}{2}\sum_{k=1}^{3} \epsilon_{kjm}\epsilon_{kin}w^i w^n \tag{A.6}$$

Furthermore, from (A.2) and (A.1) it follows that

$$dw_{jk} = \epsilon_{njk}\, dw^n = \tfrac{1}{2}\epsilon_{kjm}\epsilon^k_{in}w^i w^n$$

which, combined with (A.6), results in

$$dw_{jk} = \sum_{m=1}^{3} w_{jm}w_{mk} \tag{A.7}$$

which is the desired result (2.5).

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